String-coupled pendulum oscillators: Theory and experiment

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A coupled-oscillator system is given which is readily set up using only household materials. The normal-mode analysis of this system is worked out, and an experiment or demonstration is recommended in which one verifies the theory by measuring two times and four lengths.

There is a definite fascination to coupled oscillations, but it is hard to find a mechanical system which is both easy to build and easy to analyze. A system of one-dimensional alternating springs and masses, for example, (three springs and two masses) is easy to analyze, but not easy to build. The coupled-oscillation system of Fig. 1, however, is trivial to construct, and not too hard to analyze. It is made from light string or thread, and two equal masses (any small household object of 20 g or more will do nicely). As long as all lengths are at least 10 cm, torsion in the string will be negligible, and the theory given below will be accurate. The intended oscillations of this system are in and out of the plane of the paper.

The experiment itself is simple because a comparison to the theory requires measuring only two times and four lengths; the time T for a single energy transfer, the period T_p of pendulum-mode motion (both pendula swing together), and the lengths l_x, l_z, L, and d. The author has had his junior mechanics students build, measure, and analyze this system for the past two years, as part of a homework assignment on coupled oscillations. Most students did the work at home or in their rooms, although a few elected to set up the experiment at school. Each selected his own scale and masses, with results typically agreeing with the theory to about 5%.

The analysis begins by assuming negligible mass at points p_1 and p_2 in Fig. 1(b), so that the sum of all forces acting at point p_1 will be zero, and the same will be true at p_2. These forces are tensions, and each is parallel or antiparallel to one of the length vectors in Figure 1(b). The tensions are therefore proportional to length vectors, the coefficients having dimensions of force and length. Force equations at p_1 and p_2 are now written

$$-t_1\mathbf{d}_1 + t\mathbf{d} + T_1\mathbf{L}_1 = 0,$$

and

$$t_2\mathbf{d}_2 - t\mathbf{d} + T_2\mathbf{L}_2 = 0. \qquad (1)$$

When the system is not in motion, these equations are easily solved:

$$T_1 L = T_2 L = mg, \text{ so } T_1 = T_2,$$

and

$$t_1 l_z = T_1 L = t_2 l_z, \text{ so } t_1 = t_2,$$

and also

$$t_1 l_x = td. \qquad (2)$$

All coefficients are thus known in terms of mg and lengths. In the rest of the analysis, they are presumed to stay close to these values, even when there is small-amplitude motion.

The origin of coordinates is now chosen at P in Fig. 1(b), so that the vector from P to the center of mass of one pendulum is $\mathbf{d}_1 + \mathbf{L}_1$, and the vector from P to the other is $\mathbf{D} - \mathbf{d}_2 + \mathbf{L}_2$. Newton's second law for each pendulum is then

$$m\mathbf{g} - T_1\mathbf{L}_1 = m(\ddot{\mathbf{d}}_1 + \ddot{\mathbf{L}}_1),$$

and

$$m\mathbf{g} - T_2\mathbf{L}_2 = m(-\ddot{\mathbf{d}}_2 + \ddot{\mathbf{L}}_2). \qquad (3)$$

Adding Eqs. (3) gives

$$2\,m\mathbf{g} - T_1(\mathbf{L}_1 + \mathbf{L}_2) = m(\ddot{\mathbf{d}}_1 - \ddot{\mathbf{d}}_2 + \ddot{\mathbf{L}}_1 + \ddot{\mathbf{L}}_2). \quad (4)$$

We eliminate the $\ddot{\mathbf{d}}_1 - \ddot{\mathbf{d}}_2$ term from (4) by adding Eqs. (1) and differentiating twice with respect to the time. This results in

$$2\,m\mathbf{g} - T_1(\mathbf{L}_1 + \mathbf{L}_2) = m\,(1 + T_1/t_1)(\ddot{\mathbf{L}}_1 + \ddot{\mathbf{L}}_2). \quad (5)$$

We release the pendula of Fig. 1(a) to move primarily perpendicular to the plane of the page, so that both L_1 and L_2 should have oscillating y components. Taking the y component of (5) shows this to be the case for the sum of their y components, $Q_1 = L_{1y} + L_{2y}$:

$$\ddot{Q}_1 + w_1^2 Q_1 = 0, \qquad (6)$$

where

$$w_1^2 = \frac{T_1}{m}\left(1 + \frac{T_1}{t_1}\right)^{-1} = \frac{g}{L(1 + l_z/L)} = \frac{g}{(l_z + L)}.$$

Equation (6) is the familiar oscillator equation, and w_1 is a familiar oscillation frequency: pendula swinging together about line PP' have an effective length of $l_z + L$, and a circular frequency $w_1^2 = g/(l_z + L)$. Q_1 can thus be claimed as one of the normal modes of this system.

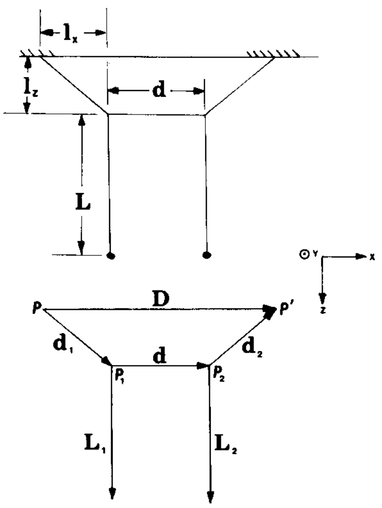

Fig. 1. (a) String-coupled pendulum oscillators. (b) Vector diagram.

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We anticipate the second normal mode as $Q_2 = L_{1y} - L_{2y}$, since we have just noted $L_{1y} = L_{2y}$ when only mode 1 is active (and so Q_2 must be zero during this time). This leads one to subtract Eqs. (3) and get an equation in the quantity $\mathbf{L}_1 - \mathbf{L}_2$, which also contains a quantity $(\mathbf{d}_1 + \mathbf{d}_2)$. This latter combination can be written in terms of a single variable using the condition that $\mathbf{d}_1 + \mathbf{d} + \mathbf{d}_2 = \mathbf{D}$:

$$(\ddot{\mathbf{d}}_2 + \ddot{\mathbf{d}}_1) = -\ddot{\mathbf{d}}. \tag{7}$$

And the single variable $\ddot{\mathbf{d}}$, in turn, can be eliminated in favor of $\mathbf{L}_1 - \mathbf{L}_2$ by using Eqs. (1) and (7) to yield a final oscillator equation:

$$-T_1(\mathbf{L}_1 - \mathbf{L}_2) = m\left(1 + \frac{T_1}{(t_1 + 2t)}\right)(\ddot{\mathbf{L}}_1 - \ddot{\mathbf{L}}_2).$$

When only this mode is active, $Q_1 = 0$, and $L_{1y} = -L_{2y}$, so that the pendula are oscillating 180° out of phase. The frequency of this motion is

$$w_2^2 = \frac{T_1}{m[1 + T_1/(2t + t_1)]} = \frac{w_1^2}{1 - x},$$

where

$$\frac{1}{x} = \left(1 + \frac{L}{l_z}\right)\left(1 + \frac{d}{2l_x}\right).$$

When we start the system from rest, with one pendulum displaced, and the other hanging vertically, both modes are present in equal amounts. As standard treatments show,[1] the "frequency" for energy transfer is half the difference in the mode frequencies. The time T for a single energy transfer is one quarter of the "period" for energy transfer, so we relate this and the mode 1 period T_p by

$$T_p/T = 2[(1 - x)^{-1/2} - 1] = x \quad (\text{for } x \ll 1). \tag{8}$$

Thus, the ratio of times depends only on x, which is calculated from two length ratios.

Equation (8) has been borne out to better than 5%, and within experimental accuracy in a number of cases. The one place where it failed, however, was the first system measured: a desktop "toy" which inspired the analysis. The calculated coupling was too low, and apparently the little pendula were so close (4 cm) that the string provided torsional as well as tensional coupling.

[1] J. B. Marion, *Classical Dynamics of Particles and Systems* (Academic, New York, 1970), pp. 408–418.